

# SOME BOUNDS FOR THE PSEUDOCHARACTER OF THE SPACE $C_\alpha(X, Y)$

ÇETIN VURAL

ABSTRACT. Let  $C_\alpha(X, Y)$  be the set of all continuous functions from  $X$  into  $Y$  endowed with the set-open topology where  $\alpha$  is a hereditarily closed, compact network on  $X$ . We obtain that:

$$\begin{aligned} i-) \quad \psi(f, C_\alpha(X, Y)) &\leq w\alpha c(X) \cdot \sup_{A \in \alpha} (\psi(f(A), Y)) \cdot \sup_{A \in \alpha} (w(f(A))) \\ ii-) \quad \psi(f, C_\alpha(X, Y)) &\leq w\alpha c(X) \cdot psw_e(f(X), Y). \end{aligned}$$

## 1. INTRODUCTION AND TERMINOLOGY

Let  $X$  and  $Y$  be topological spaces, and let  $C(X, Y)$  denote the set of all continuous mappings from  $X$  into  $Y$ . Let  $\alpha$  be a collection of subsets of  $X$ . The topology having subbase  $\{[A, V] : A \in \alpha \text{ and } V \text{ is an open subset of } Y\}$  on the set  $C(X, Y)$  is denoted by  $C_\alpha(X, Y)$  where  $[A, V] = \{f \in C(X, Y) : f(A) \subseteq V\}$ . If  $\alpha$  consists of all finite subsets of  $X$ , then the set  $C(X, Y)$  endowed with that topology is called pointwise convergence topology and denoted by  $C_p(X, Y)$ .

The cardinality and the closure of a set is denoted by  $|A|$  and  $cl(A)$ , respectively. The restriction of a mapping  $f : X \rightarrow Y$  to a subset  $A$  of  $X$  is denoted by  $f|_A$ .  $T(X)$  denotes the set of all non-empty open subsets of a topological space  $X$ .  $ord(x, \mathcal{A})$  is the cardinality of the collection  $\{A \in \mathcal{A} : x \in A\}$ . Throughout this paper  $X$  and  $Y$  are regular topological spaces, and  $\alpha$  is a hereditarily closed, compact network on the domain space  $X$ . (i.e.,  $\alpha$  is a network on  $X$  such that each member of it is compact and each closed subset of a member of it is a member of  $\alpha$ .) Without loss of generality, we may assume that  $\alpha$  is closed under finite unions. Recall that the *weak  $\alpha$ -covering number* of  $X$  is defined to be  $w\alpha c(X) = \min\{|\beta| : \beta \subseteq \alpha \text{ and } \bigcup \beta \text{ is dense in } X\}$ . The *weight*, *density* and *character* of a space  $X$  are denoted by  $w(X)$ ,  $d(X)$  and  $\chi(X)$ , respectively. The  *$i$ -weight* of a topological space  $X$ , is the least of cardinals  $w(Y)$  of the Tychonoff spaces  $Y$  which are continuous one-to-one images of  $X$ . The *pseudocharacter of a space  $X$  at a subset  $A$* , denoted by  $\psi(A, X)$ , is defined as the smallest cardinal number of the form  $|\mathcal{U}|$ , where  $\mathcal{U}$  is a family of open subsets of  $X$  such that  $\bigcap \mathcal{U} = A$ . If  $A = \{x\}$  is a singleton, then we

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2010 *Mathematics Subject Classification.* 54C35, 54A25, 54C05.

*Key words and phrases.* Pseudocharacter, function space.

The author acknowledge Mr. Hasan Gül for the first draft of manuscript.

write  $\psi(x, X)$  instead of  $\psi(\{x\}, X)$ . The pseudocharacter of a space  $X$  is defined to be  $\psi(X) = \sup\{\psi(x, X) : x \in X\}$ . The diagonal number  $\Delta(X)$  of a space  $X$  is the pseudocharacter of its square  $X \times X$  at its diagonal  $\Delta_X = \{(x, x) : x \in X\}$ .

The pseudocharacter of the space  $C(X, Y)$  has been studied, and some remarkable equalities or inequalities was obtained between the pseudocharacter of the space  $C(X, Y)$  for certain topologies and some cardinal functions on the spaces  $X$  and  $Y$ . For instance, in [3], the inequalities  $\psi(Y) \leq \psi(C_p(X, Y)) \leq \psi(Y) \cdot d(X)$  and, in [1] and [2], the equalities  $\psi(C_p(X, \mathbb{R})) = d(X) = iw(C_p(X, \mathbb{R}))$ , and in [6],  $\psi(C_\alpha(X, \mathbb{R})) = \Delta(C_\alpha(X, \mathbb{R})) = wac(X)$  were obtained. In this paper, when the range space  $Y$  is an arbitrary topological space instead of the space  $\mathbb{R}$ , we obtained some inequalities between the pseudocharacter of the space  $C_\alpha(X, Y)$  at a point  $f$  and the weak  $\alpha$ -covering number of the domain space  $X$  and some cardinal functions on the range space  $Y$ .

We assume that all cardinal invariants are at least the first infinite cardinal  $\aleph_0$ .

Notations and terminology not explained above can be found in [4] and [5].

## 2. MAIN RESULTS

First, we give an inequality between the pseudocharacter of a point  $f$  in the space  $C_\alpha(X, Y)$  and some cardinal functions on spaces  $X$  and  $Y$ .

**Theorem 2.1.** *For each  $f \in C_\alpha(X, Y)$ , we have*

$$\psi(f, C_\alpha(X, Y)) \leq wac(X) \cdot \sup_{A \in \alpha} (\psi(f(A), Y)) \cdot \sup_{A \in \alpha} (w(f(A)))$$

*Proof.* Let  $wac(X) \cdot \sup\{\psi(f(A), Y) : A \in \alpha\} \cdot \sup\{w(f(A)) : A \in \alpha\} = \kappa$ . The inequality  $wac(X) \leq \kappa$  gives us a subfamily  $\beta = \{A_i : i \in I\}$  of  $\alpha$  such that  $|I| \leq \kappa$  and  $X = cl(\bigcup \beta) = cl(\bigcup \{A_i : i \in I\})$ . Since  $\psi(f(A_i), Y) \leq \kappa$  for each  $i \in I$ , there exists a family  $\mathcal{V}_i$  consisting of open subsets of the space  $Y$  such that  $|\mathcal{V}_i| \leq \kappa$  and  $f(A_i) = \bigcap \{V : V \in \mathcal{V}_i\}$  for each  $i \in I$ . Since  $w(f(A_i)) \leq \kappa$  for each  $i \in I$ , the subspace has a base  $\mathcal{B}_i$  with  $|\mathcal{B}_i| \leq \kappa$ . For each  $i \in I$ , let

$$\mathcal{H}_i = \{[A_i \cap f^{-1}(cl(G)), Y \setminus cl(U)] : G, U \in \mathcal{B}_i \text{ and } cl(G) \cap cl(U) = \emptyset\},$$

$$\mathcal{R}_i = \{[A_i, V] : V \in \mathcal{V}_i\} \text{ and } \mathcal{W} = (\bigcup_{i \in I} \mathcal{R}_i) \cup (\bigcup_{i \in I} \mathcal{H}_i).$$

It is clear that  $|\mathcal{W}| \leq \kappa$  and  $f \in W$  for each  $W \in \mathcal{W}$ . Now, we shall prove that  $\bigcap \mathcal{W} = \{f\}$ . Take a  $g \in \bigcap \mathcal{W}$ . We claim that  $g|_{A_i} = f|_{A_i}$  for each  $i \in I$ . Assume the contrary. Suppose  $g|_{A_j} \neq f|_{A_j}$  for some  $j \in I$  that is, we have an  $x \in A_j$  such that  $f(x) \neq g(x)$ . Since  $g \in \bigcap \mathcal{W}$  and  $f(A_j) = \bigcap \{V : V \in \mathcal{V}_j\}$ , we have  $g(A_j) \subseteq f(A_j)$ . Therefore  $g(x) \in f(A_j)$  and  $f(x) \in f(A_j)$ . Since  $f(x) \neq g(x)$  and the space  $Y$  is regular, there exist  $G$  and  $U$  in  $\mathcal{B}_j$  such that  $f(x) \in cl(G)$ ,  $g(x) \in cl(U)$  and  $cl(G) \cap cl(U) = \emptyset$ . On the other hand, since  $[A_j \cap f^{-1}(cl(G)), Y \setminus cl(U)] \in \mathcal{H}_j$  and

$g \in \bigcap \mathcal{W}$ , we have  $g \in [A_j \cap f^{-1}(cl(G)), Y \setminus cl(U)]$ . But this contradicts to the fact that  $g(x) \in cl(U)$ . Hence,  $g|_{A_i} = f|_{A_i}$  for each  $i \in I$ , or in other words  $g|_{\bigcup_{i \in I} A_i} = f|_{\bigcup_{i \in I} A_i}$ . Hausdorffness of the space  $Y$  and the equality  $X = cl(\bigcup \beta) = cl(\bigcup \{A_i : i \in I\})$  lead us to the fact that  $g = f$ . Therefore  $\bigcap \mathcal{W} = \{f\}$ , that is  $\psi(f, C_\alpha(X, Y)) \leq \kappa$ .  $\square$

Recall that a cover  $\mathcal{A}$  of a set  $X$  is called a *separating cover* if  $\bigcap \{A \in \mathcal{A} : x \in A\} = \{x\}$ , for each  $x \in X$ . Also recall that the *point separating weight*  $psw(X)$  of a topological space  $X$  is the smallest infinite cardinal  $\kappa$  such that the space  $X$  has a separating open cover  $\mathcal{V}$  with  $ord(x, \mathcal{V}) \leq \kappa$  for each  $x \in X$ .

**Definition 2.2.** Let  $A$  be a subset of a topological space  $(X, \tau)$ . We say that the *point separating exterior weight*  $psw_e(A, X) \leq \kappa$ , if there exists a subfamily  $\mathcal{V} \subseteq \tau$  satisfying  $ord(a, \mathcal{V}) \leq \kappa$  and  $\bigcap \{V \in \mathcal{V} : a \in V\} = \{a\}$ , for each  $a \in A$ .

The following lemmas are needed for the second main theorem, and in order to prove them, let us recall the Miščenko's lemma.

**Lemma 2.3 (Miščenko's lemma [5]).** Let  $\kappa$  be an infinite cardinal, let  $X$  be a set, and let  $\mathcal{A}$  be a collection of subsets of  $X$  such that  $ord(x, \mathcal{A}) \leq \kappa$  for all  $x \in X$ . Then the number of finite minimal covers of  $X$  by elements of  $\mathcal{A}$  is at most  $\kappa$ .

**Lemma 2.4.** Let  $Z$  be subspace of the space  $X$  such that  $psw_e(Z, X) \leq \kappa$ . Then  $\psi(K, X) \leq \kappa$  for each compact subset  $K$  of  $Z$ .

*Proof.* Let  $\mathcal{V}$  be a family of open subsets of  $X$  satisfying  $ord(z, \mathcal{V}) \leq \kappa$  and  $\bigcap \{V \in \mathcal{V} : z \in V\} = \{z\}$ , for each  $z \in Z$ . Let  $K$  be any compact subspace of  $Z$  and let

$\mu = \{\mathcal{W} : \mathcal{W} \subseteq \mathcal{V} \text{ and } \mathcal{W} \text{ is a minimal finite open cover for } K\}$ . By Miščenko's lemma, we have  $|\mu| \leq \kappa$ .

Define the family  $\mathcal{O} = \{\bigcup_{W \in \mathcal{W}} W : \mathcal{W} \in \mu\}$ . It is clear that  $|\mathcal{O}| \leq \kappa$  and it can be easily seen that  $\bigcap \mathcal{O} = K$ . Hence  $\psi(K, X) \leq \kappa$ .  $\square$

**Lemma 2.5.** Let  $Z$  be subspace of the space  $X$  such that  $psw_e(Z, X) \leq \kappa$ . Then we have  $w(K) \leq \kappa$  for each compact subset  $K$  of  $Z$ .

*Proof.* Let  $K$  be a compact subset of  $Z$ . Clearly,  $psw(K) \leq psw(Z) \leq psw_e(Z, X)$ . The compactness of  $K$  leads us to the fact that  $w(K) = psw(K)$ . [in [5], Ch. 1, Theorem 7.4]. Hence the claim.  $\square$

Now, we are ready to give another bound for the pseudocharacter of the space  $C_\alpha(X, Y)$  at a point  $f$ .

**Theorem 2.6.** For each  $f \in C_\alpha(X, Y)$ , we have

$$\psi(f, C_\alpha(X, Y)) \leq wac(X) \cdot psw_e(f(X), Y).$$

*Proof.* Let  $w\alpha c(X) \cdot psw_e(f(X), Y) = \kappa$ , and let  $\beta = \{A_i : i \in I\}$  be a subfamily of  $\alpha$  such that  $cl(\bigcup \beta) = X$  and  $|I| \leq \kappa$ . The compactness of  $A_i$  for each  $i \in I$  and the inequality  $psw_e(f(X), Y) \leq \kappa$  lead us to the facts that  $\psi(f(A_i), Y) \leq \kappa$  and  $w(A_i) \leq \kappa$  for each  $i \in I$ , by lemmas.2.4 and 2.5. Therefore, by Theorem 2.1, we have  $\psi(f, C_\alpha(X, Y)) \leq \kappa$ .  $\square$

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GAZI UNIVERSITESI, FEN FAKULTESI, MATEMATİK BOLUMU, 06500 TEKNİKOKULLAR, ANKARA, TURKEY

*E-mail address:* cvural@gazi.edu.tr